НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ УНИВЕРСИТЕТ

Существенное множество: свойства и практическое применение

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Alternatives, comparisons, choices

X – the *general set* of alternatives.

A – the *feasible set* of alternatives: $A \subseteq X \wedge A \neq \varnothing \wedge |A| < \infty$. The feasible set is a variable.

R – results of binary comparisons, $R \subseteq X \times X$.

R is presumed to be complete: $\forall\, x \in X, \forall\, y \in X, (x, y) \in R \vee (y, x) \in R$.

$R|_A = R \cap A \times A$ – restriction of R onto A.

$(A, R|_A)$ – *abstract game or weak tournament.*

P – asymmetric part of R, $P \subseteq R$: $(x, y) \in P \Leftrightarrow ((x, y) \in R \wedge (y, x) \notin R)$.

If $P|_A$ is complete, $\forall\, x \in X, \forall\, y \in X \wedge y \neq x, (x, y) \in P \vee (y, x) \in P$, then

$(A, R|_A)$ – (*proper*) *tournament.*

Tournament solutions

A *tournament solution S* is a choice correspondence $S(A, R)$: $2^X \backslash \varnothing \times 2^{X \times X} \to 2^X$

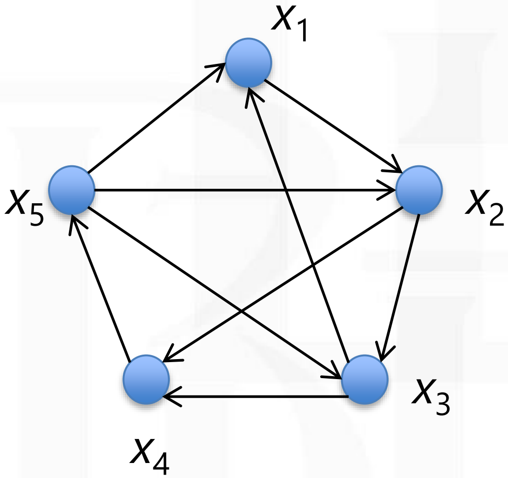
that has the following properties:

0. Locality: $S(A, R) = S(R|_A) \subseteq A$

1. *Nonemptiness*: $\forall A, \forall R, \; S(R|_A) \neq \varnothing$;
2. *Neutrality*: permutation of alternatives' names and choice commute;
3. *Condorcet consistency*: $MAX(R|_A) \subseteq S(R|_A) \wedge MAX(R|_A) = \{w\} \Rightarrow S(R|_A) = \{w\}$.

	x_1	x_2	x_3	x_4	x_5
x_1	0.5	1	0	0.5	0
x_2	0	0.5	1	1	0
x_3	1	0	0.5	1	0
x_4	0.5	0	0	0.5	1
x_5	1	1	1	0	0.5

Tournament matrix **T**



Tournament digraph

Tournament game (TG)

TG is a two-player zero-sum symmetric non-cooperative game on a tournament $R|_A$

Set of players $N=\{1, 2\}$. *Sets of pure strategies* $S_1=S_2=A$. *Payment functions*:

$v_1(x_1, x_2)=1 \Leftrightarrow x_1Px_2$, $v_1(x_1, x_2)=-1 \Leftrightarrow x_2Px_1$, $v_1(x_1, x_2)=0$ otherwise, $v_2(x_1, x_2) = -v_1(x_1, x_2)$.

TG has Nash equilibria in pure strategies $\Leftrightarrow MAX(R|_A)\neq\emptyset$.

A mixed strategy in TG is a lottery $\mathbf{p}$ on A. Then $v_1(\mathbf{p}_1, \mathbf{p}_2) = \mathbf{p}_1\mathbf{G}\mathbf{p}_2$,

where matrix $\mathbf{G}$ is obtained from the tournament matrix $\mathbf{T}$: $g_{ij} = 2t_{ij}-1$.

$(v_1(\mathbf{p}_1, \mathbf{p}_2)+1)/2=\mathbf{p}_1\mathbf{T}\mathbf{p}_2$ is the probability that player 1 will win the game.

Since $\mathbf{G}$ is antisymmetric, formula $\mathbf{p}_1\mathbf{G}\mathbf{p}_2$ defines a binary relation on the set of lotteries:

$$\mathbf{p}_1\mathbf{G}\mathbf{p}_2 \geq 0 \Leftrightarrow \mathbf{p}_1 \gtrsim \mathbf{p}_1$$

If $\mathbf{p}_0\mathbf{G}\mathbf{p} \geq 0$ for all $\mathbf{p}$ then $\mathbf{p}_0$ is a ***maximal lottery***.

$\mathbf{p}_1$ and $\mathbf{p}_2$ are maximal lotteries $\Leftrightarrow (\mathbf{p}_1, \mathbf{p}_2)$ is a Nash equilibrium of TG

Bipartisan set (BP) and Essential set (ES)

Theorem:

1. The set of maximal lotteries is always nonempty.
2. If a tournament $(A, R|_A)$ is proper then there is just one maximal lottery.

Bipartisan set *BP* (Laffond, Laslier, Le Breton, 1993)

of a (proper) tournament $(A, R|_A)$ is the support of the maximal lottery.

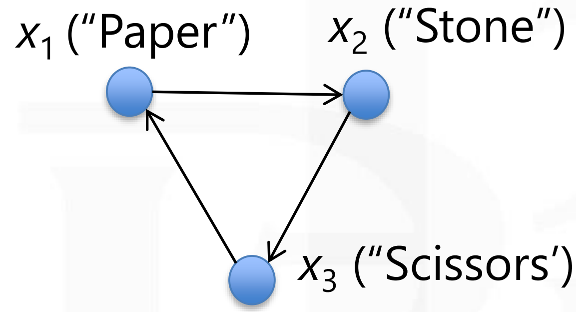
Essential set *E* (Dutta, Laslier, 1999)

of a (weak) tournament $(A, R|_A)$ is the union of supports of all maximal lotteries.

Example

Tournament digraph – the Condorcet cycle.

$A=\{x_1, x_2, x_3\}$, $R|_A=\{(x_1, x_2), (x_1, x_2), (x_1, x_2)\}$



	x_1	x_2	x_3
x_1	0.5	1	0
x_2	0	0.5	1
x_3	1	0	0.5

Tournament matrix **T**

	x_1	x_2	x_3
x_1	0	1	-1
x_2	-1	0	1
x_3	1	-1	0

Matrix **G**

Tournament game – "Paper, Scissors, Stone".

$MAX(R|_A)=\varnothing \Rightarrow$ no Nash equilibrium in pure strategies.

Maximal lottery $\mathbf{p}_{max}$= (1/3, 1/3, 1/3).

Bipartisan set $BP = A$.

Note that $\mathbf{p}_{max}$ is an eigenvector of **G** with the eigenvalue 0, therefore $\mathbf{pGp}_{max}=0$ for all $\mathbf{p}$.

Properties

- ***Monotonicity*** (*monotonicity w.r.t. results of binary comparisons*):

  $\forall R_1, R_2 \subseteq X^2, \forall A \subseteq X, \forall x \in S(R_1|_A),$

  $(R_1|_{A\setminus\{x\}} = R_2|_{A\setminus\{x\}} \wedge \forall y \in A\setminus\{x\}, (xP_1y \Rightarrow xP_2y) \wedge (xR_1y \Rightarrow xR_2y)) \Rightarrow x \in S(R_2|_A).$

- ***Stability***

  For all $R \subseteq X^2$ and for all $A, B \subseteq X$ such that $A \cap B \neq \varnothing$ the following holds:

  $$S(A, R) = S(B, R) = C \Leftrightarrow S(A \cup B, R) = C.$$

- ***Computational simplicity***: There is a polynomial algorithm for computing S.

Properties related to stability

Stability: $S(A, R)=S(B, R)=C \Leftrightarrow S(A\cup B, R)=C$.

- ***α-property*** (*generalized Nash independence of irrelevant alternatives, independence of outcasts, strong superset property*):

$$S(A, R)=S(B, R)=C \Leftarrow S(A\cup B, R)=C.$$

- ***γ-property***:

$$S(A, R)=S(B, R)=C \Rightarrow S(A\cup B, R)=C.$$

- ***Idempotency***: $\forall A,\ S(S(A))=S(A)$.
- ***The Aizerman-Aleskerov condition***: $\forall A, \forall B,\ S(A)\subseteq B\subseteq A \Rightarrow S(B)\subseteq S(A)$.
- ***Independence of irrelevant results*** (*independence of losers*):

$\forall R_1,R_2\subseteq X^2, \forall A\subseteq X, (\forall x\in S(R_1|_A),\forall y\in A, ((xR_1y \Leftrightarrow xR_2y) \wedge (yR_1x \Leftrightarrow yR_2x)) \Rightarrow S(R_1|_A)=S(R_2|_A)$.

α-property ⇔ Idempotency ∧ the Aizerman-Aleskerov condition

α-property ⇒ Independence of irrelevant results

Axiomatic analysis

	BP	*E*
Monotonicity	Yes	Yes
α-property (outcast)	Yes	Yes
Idempotence	Yes	Yes
Aizerman-Aleskerov property	Yes	Yes
Independence of irrelevant results	Yes	Yes
γ-property	Yes	Yes
Stability	Yes	Yes
Computational simplicity	Yes	Yes

The covering relations and the uncovered sets

The covering relations (Fishburn, 1977; Miller, 1980)

The covering relation $C \subseteq A^2$, is a strengthening of $P|_A$:

1. The Miller covering C_M: $xC_My \Leftrightarrow xPy \wedge P^{-1}(y) \subset P^{-1}(x)$.
2. The Fishburn covering C_F: $xC_Fy \Leftrightarrow xPy \wedge P(x) \subset P(y)$.
3. The McKelvey covering C_{McK}: $xC_{McK}y \Leftrightarrow xPy \wedge P^{-1}(y) \subset P^{-1}(x) \wedge P(x) \subset P(y)$.

The set of all alternatives that are not covered in A by any alternative is called ***the uncovered set*** of a feasible set A.

The Miller, Fishburn and McKelvey uncovered sets will be denoted UC_M, UC_F and UC_{McK}, correspondingly.

Minimal externally stable sets

A nonempty subset *B* of *A* is called

P-dominating	if	$\forall x \in A$,	$\exists y \in B$: yPx
P-externally stable	if	$\forall x \in A \backslash B$,	$\exists y \in B$: yPx
R-externally stable	if	$\forall x \in A \backslash B$,	$\exists y \in B$: yRx
Self-protecting	if	$\forall x \in A$,	$(\exists y \in B: yPx) \vee (\forall y \in B, yRx)$
Weakly stable	if	$\forall x \in A \backslash B$,	$(\exists y \in B: yPx) \vee (\forall y \in B, yRx)$

Tournament solutions: the union of all minimal

P-dominating sets	*D*	(Duggan 2013, Subochev 2016)
P-externally stable sets	*ES*	(Wuffl, Feld, Owen & Grofman 1989, Subochev 2008)
R-externally stable sets	*RES*	(Aleskerov & Subochev 2009, 2013)
Self-protecting sets	*SP*	(Roth 1976, Subochev 2020)
Weakly stable sets	*WS*	(Aleskerov & Kurbanov 1999)

Relations of E to other solutions

In proper tournaments, $BP \subseteq UC \subseteq ES$, also $BP \subseteq D \subseteq ES$.

In weak tournaments,

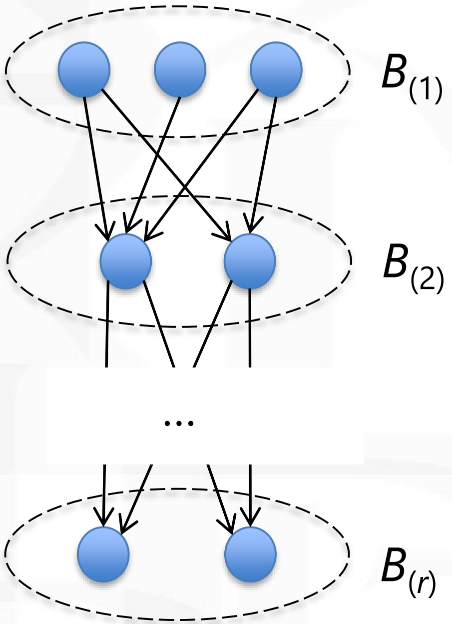
1. $E \subseteq UC_{\text{McK}}$ (Dutta, Laslier, 1999)
2. $E \not\subset UC_{\text{M}} \wedge UC_{\text{M}} \not\subset E$, it remains to be proven that $E \cap UC_{\text{M}} \neq \varnothing$ always holds.
3. $E \not\subset UC_{\text{F}} \wedge UC_{\text{F}} \not\subset E$, it remains to be proven that $E \cap UC_{\text{M}} \neq \varnothing$ always holds.
4. $E \not\subset ES \wedge ES \not\subset E$, but $E \cap ES \neq \varnothing$ always holds.
5. $E \not\subset D \wedge D \not\subset E$, but $E \cap D \neq \varnothing$ always holds.
6. $RES \not\subset E$ and $E \cap RES \neq \varnothing$ always holds.
7. $E \not\subset SP \wedge SP \not\subset E$, but $E \cap SP \neq \varnothing$ always holds.
8. $E \not\subset WS \wedge WS \not\subset E$, but $E \cap WS \neq \varnothing$ always holds.

Ranking based on a tournament solution

Suppose, we are interested in ranking alternatives from A.

Then we may use the following procedure:

- Tournament solution $S(A, R)$ choses the set $B_{(1)}$ of the best alternatives in A, $B_{(1)}=S(A, R)$.
- Exclude these alternatives from A and apply S to the rest. $B_{(2)}=S(A\backslash B_{(1)}, R)=S(A\backslash S(A, R), R)$ will be the set of the second-best alternatives in A.
- By repeated exclusion of the best alternatives determined at each step of the procedure the set A is separated into groups $B_{(r)}=S(A\backslash(B_{(r-1)}\cup B_{(r-2)}\cup\ldots\cup B_{(2)}\cup B_{(1)}), R)$, and that is the ranking.
- Let r denote the rank of x in this ranking.



Bibliometric data

Indicator	*Database*	*Year*	*Publication window, years*	*Weighted*	*Size-dependent*
impact factor	WoS/JCR	2011	2	No	No
5-year impact factor	WoS/JCR	2011	5	No	No
immediacy index	WoS/JCR	2011	1	No	No
article influence	WoS/JCR	2011	5	Yes	No
Hirsch index	WoS	2007–2011 (papers and citations)	5	No	Yes
SNIP	Scopus	2011	3	No	No
SJR	Scopus	2011	3	Yes	No

Economics: 212 journals
Management: 93
Political Science: 99

Severity of Condorcet paradox evaluated

Numbers of 3-, 4- and 5-step *P*-cycles and ties

	3-step cycles	***4-step cycles***	***5-step cycles***	***Tied pairs***	***All pairs***
Economics	2446	22427	226103	197	22366
Management	203	787	3254	33	4278
Political Science	149	430	1344	73	4851

Discrimination

Total numbers of ranks in rankings based on sorting

	Number of journals	UC_M	*ES*	*E*
Management	93	42	33	49
Political Science	99	42	36	45

Formal analysis of correlations

Kendall τ_b (economic journals)

	IF	5-IF	Immediacy	AI	Hirsch	SNIP	SJR
5-year IF	**0.830**	**1.000**	0.510	0.725	0.702	0.726	0.741
Markovian	0.819	0.891	**0.560**	**0.769**	**0.729**	**0.750**	**0.775**

The Markovian ranking majority-dominates the ranking based on 5-IF

ERGO

The Markovian ranking represents the set of seven single-indicator-based rankings better than the ranking based of 5-year impact factor

The rankings of rankings (based on τ_b)

rank	Management	Political Science
1	*E*	*ES*
2	*ES*	UC_M
3	UC_M	Copeland 3
4	Copeland 2	Copeland 2
5	Copeland 3	*E*
6	Markovian	Markovian
7	5-IF	5-IF
8	SNIP	Hirsch
9	Hirsch	AI / IF / SJR
10	AI	
11	SJR	
12	IF	SNIP
13	Immediacy	Immediacy

The rankings of rankings (based on the share of strictly coinciding pairs)

rank	Managemen t	Political Science
1	Copeland 3	Copeland 2 / Copeland 3 / Markovian
2	Copeland 2	
3	Markovian	
4	*E*	*E*
5	UC_M	UC_M
6	5-IF	5-IF
7	*ES*	*ES*
8	AI	SNIP
9	IF	AI
10	SNIP	IF / Hirsch / SJR
11	SJR	
12	Hirsch	
13	Immediacy	Immediacy

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Спасибо
за внимание!

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